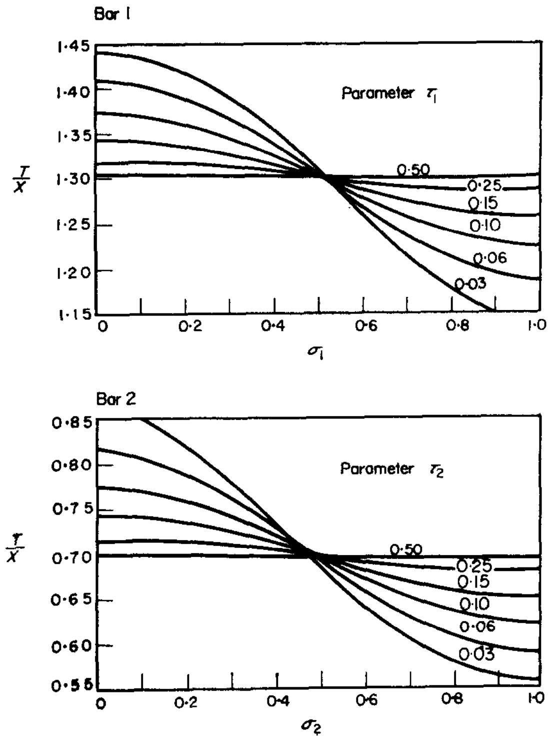

FIG. 2. Temperature distribution represented by equations (3) and (4); $\tau_0 = 0{\cdot}10$.

The equations derived are equally valid for molecular diffusion.

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THE TRANSITION FROM BLACK BODY TO ROSSELAND FORMULATIONS IN OPTICALLY THICK FLOWS†

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THE energy radiated by optically thick flows through their boundaries is expressed sometimes as a black body flux $q^R = \sigma T^4$, sometimes as a Rosseland diffuse flux

$$q^R = -\frac{16\sigma T^3}{3k_R}\frac{\partial T}{\partial y}.$$

The purpose of this note is to establish, on the basis of two simple incompressible flows, the radiation parameters which govern the choice between these two formulations.

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THE INVISCID INCOMPRESSIBLE PARALLEL FLOW

The first flow to be considered here is the classical inviscid incompressible medium flowing at a velocity u_∞ past the entrance of a channel or past the edge of a flat plate (Fig. 1). The conservation of energy equation simplifies in this case to:

$$\rho_\infty u_\infty \frac{\partial h}{\partial x} = -\frac{\partial q^R}{\partial y}. \tag{1}$$

After substitution of the Rosseland approximation on the right-hand side and using the relation $dh = c_p\, dT$, equation (1) becomes:

$$\rho_\infty u_\infty c_p \frac{\partial T}{\partial x} = \frac{\partial}{\partial y}\left(\frac{16\sigma T^3}{3k_R}\frac{\partial T}{\partial y}\right). \tag{2}$$

For the sake of simplicity, we now assume that the volumetric absorption coefficient k_R varies as a third power of the temperature. Although this assumption is taken only as a convenience for the mathematical treatment, it represents a qualitatively correct trend and is, at any rate, a better assumption than that of a constant coefficient k_{Ri}. If we characterize by the subscript i the initial uniform state:

$$k_R = k_{Ri}\frac{T^3}{T_i^3} \tag{3}$$

and the energy equation becomes:

$$\rho_\infty u_\infty c_p \frac{\partial T}{\partial x} = \frac{16\sigma T_i^3}{3k_{Ri}}\frac{\partial^2 T}{\partial y^2}. \tag{4}$$

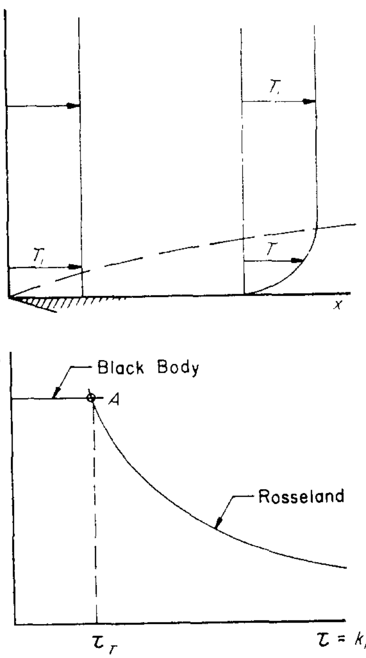

FIG. 1. Inviscid radiating flow past a flat plate.

This partial differential equation with constant coefficients presents exactly the same mathematical form as that obtained when solving for the temperature distribution across an infinitely thick plate, when a sudden temperature change occurs at the surface [1]:

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial y^2}. \tag{5}$$

The same form of solution is therefore applicable and the temperature profile and radiant heat flux can be written:

$$\frac{T - T_w}{T_i - T_w} = \text{erf}\left[\frac{y}{2\sqrt{(\Gamma_k x^2)}}\right]$$

$$q^R = -\frac{16\sigma T_i^4}{3k_{Ri}}\frac{\exp(-y^2/4\Gamma_k x^2)}{\sqrt{(\pi\Gamma_k x^2)}}. \tag{6}$$

At the wall ($y = 0$):

$$q^R(0) = -\frac{16\sigma T_i^4}{3k_{Ri}x}\frac{1}{\sqrt{(\pi\Gamma_k)}} \tag{7}$$

where:

$$\Gamma_k = \frac{16\sigma T_i^3}{3u_\infty\rho_\infty c_p}\cdot\frac{1}{k_{Ri}x} = \frac{16}{3Bo\,.\,\tau}, \tag{8}$$

$$Bo = \text{Boltzmann Number} = \frac{\rho_\infty u_\infty c_p T_i}{\sigma T_i^4} \tag{9}$$

$$\tau = \text{optical length} = k_{Ri}x. \tag{10}$$

One notes (Fig. 2) that the heat flux to the plate is inversely proportional to the square root of x so that $q^R(0)$ tends to infinity near the forward edge of the wall

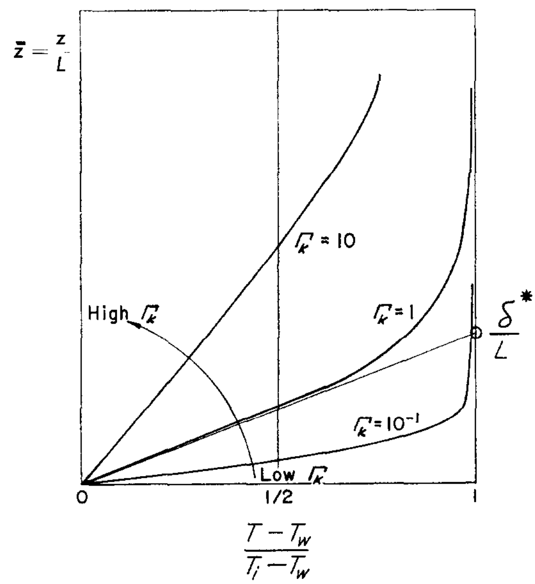

FIG. 2. Stagnation temperature profile in an inviscid radiating flow.

($x \to 0$). Clearly, the validity of the Rosseland approximation does not hold in this region $[(\partial^3 T/\partial y^3) \neq 0]$, while the black body approximation (emission from an optically thick slab of uniform temperature T_i) becomes valid at $x \simeq 0$.

To acquire an idea of the length x_T at which this transition would take place, the two expressions of the flux are equated:

$$\sigma T_i^4 = \frac{16\sigma T_i^4}{3k_{Ri}x_T}\,\frac{1}{\sqrt{(\pi\Gamma_k)}}$$

hence:

$$\tau_T \equiv k_{Ri}x_T = \frac{16}{3\pi}\,Bo. \tag{11}$$

For large Boltzmann numbers (i.e. small radiation losses in comparison to the energy stored in the fluid), τ_T will be large and the black body assumption will be satisfactory for equipment of length less than x_T.

The analogy with the drag problem in fluid mechanics near the edge of a flat plate (Maxwellian "free molecular" approximation vs. boundary-layer solution) [2] is quite apparent. In particular, a closer analysis of the radiation problem should bring out a transition regime between the black body and the Rosseland approximations in analogy with the "slip flow" of rarefied gases.

Also note that the radiation mean free path $l_R \equiv 1/k_R$; the optical length $\tau = x/l_R$ plays, therefore, a role analogous to that of the Knudsen number of classical fluid dynamics: $Kn \equiv l_c/x$, l_c being the mean molecular free path.

THE INVISCID INCOMPRESSIBLE STAGNATION FLOW

Another incompressible flow of interest is the potential flow in a stagnation area; in this case [3]:

$$u = ax, \qquad w = -2az \tag{12}$$

where a is some characteristic† velocity gradient V/L.

The energy equation is therefore of the form:

$$2az\rho c_p \frac{\partial T}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{16\sigma T^3}{3k_R}\frac{\partial T}{\partial z}\right). \tag{13}$$

If we assume that the volumetric absorption coefficient k_R varies as a cubic power of the temperature and that all other thermal properties are constant, this equation reduces to the form:

$$2z\frac{\partial T}{\partial z} = -L^2\Gamma_k\frac{\partial^2 T}{\partial z^2} \tag{14}$$

where:

$$\Gamma_k = \frac{16\sigma T_i^3}{3V\rho c_p}\,\frac{1}{k_{Ri}L} = \frac{16}{3Bo\,.\,\tau_L}. \tag{15}$$

† Some typical stagnation flows (3) are that of a sphere $[a = (3V/2R)$, hence $L = (2R/3)]$ and of a cylinder of axis normal to the flow $[a = (2V/R)$, hence $L = (R/2)]$ (V is the upstream velocity at infinity, R is the radius).

The solution of this ordinary differential equation in $\partial T/\partial z$ yields at the wall:

$$q^R(0) = 2\frac{16\sigma T_i^4}{3k_{Ri}L}\,\frac{T_w/T_i - 1}{\sqrt{(\pi\Gamma_k)}}. \tag{16}$$

This solution is identical, for $T_w = 0$ to that of the flux to a flat plate [equation (7)] and the same conclusions hold about the range of validity of the Rosseland and black body approximations.

It is interesting to consider in the second case the temperature profile near the wall. If we introduce the nondimensional variables $\bar{T} = T/T_i$ and $\bar{z} = z/L$, the solution of equation (14) is:

$$\frac{\bar{T} - 1}{\bar{T}_w - 1} = \operatorname{erfc}\,[\sqrt{(1/\Gamma_k)}\bar{z}] \tag{17}$$

where erfc is the complementary error function:

$$\operatorname{erfc}(x) \equiv 1 - \operatorname{erf}(x) \equiv 1 - \frac{2}{\sqrt{\pi}}\int_0^x \exp(-\xi^2)\,d\xi. \tag{18}$$

If the temperature profile is plotted near the wall (Fig. 2), the influence of an increasing Γ_k is readily seen. It illustrates the physical fact that in low Γ_k flows, the layers near the wall lose more rapidly their stored energy by radiation than for higher values of Γ_k. At the limit $\Gamma_k \to 0$, we obtain the isothermal slab (black body radiation).

Also note that a more characteristic length δ^* of the flow can be substituted to L:

$$\delta^* \equiv L\sqrt{(\pi\Gamma_k)}. \tag{19}$$

In this case equation (16) becomes (assuming for simplicity $T_w = 0$):

$$q^R(0) = -\frac{16\sigma T^3}{3k_R}\frac{\partial T}{\partial y}\bigg|_{y=0} = -\frac{16\sigma T_i^4}{3k_{Ri}}\,\frac{1}{\delta^*} \tag{20}$$

and δ^* is truly a representative "thermal layer thickness," depending on Γ_k. If we chose to use this thickness δ^* to identify the transition "point" between black body and diffusive Rosseland approximations (point A of Fig. 1), we simply obtain:

$$\tau_T^* = k_{Ri}\delta^* = \frac{16}{3} \simeq 5. \tag{21}$$

REMARKS

1. In this treatment, the assumption of an inviscid flow was equivalent to that of no energy exchange by molecular collision; conduction was therefore neglected as well as viscous drag. In other words:

$$N_{R-c} \equiv \frac{q^R}{q_c} \gg 1.$$

The value of N_{R-c} in practical cases will naturally depend on whether q^R takes a black body value σT^4 or a Rosseland one

$$-\frac{16\sigma T^3}{3k_R}\frac{\partial T}{\partial y}$$

and also on whether q^c takes the continuous value

$$- k_c \frac{\partial T}{\partial y}$$

or a free molecular one $2\pi mkT$. The situation $N_{R-c} \gg 1$ is naturally the common rule in those engineering designs where radiation transfer is predominant (e.g. radiant boilers). A study of the case when both fluxes are differential and N_{R-c} takes a range of finite values has appeared in (4).

2. The validity of the Rosseland approximation within an optical thickness of 2 or 3 of the wall surface is just as questionable as that of molecular transfer laws within a few mean free paths of a solid surface. Corrections must be made in both cases to account for the strong non-equilibrium character of the fluid–solid interface [2, 4].

If, however, one is primarily interested in the heat transfer to the wall [and, if the optical dimensions of the system are large, ($\tau_L \gg 2$)], this local perturbation can be overlooked for the radiation case as it is classically done in conduction or viscous drag problems. Hence the use of the Rosseland approximation in this note.

CONCLUSION

On the basis of the analysis of two simple fluid flows, it is possible to characterize by dimensionless parameters the respective domains of validity of the black body and Rosseland formulations in optically thick flows.

A meaningful "radiation thermal thickness" δ^* was shown to be a function of the dimensionless radiation–convection ratio Γ_k. The analogy of this radiation transfer problem with classical viscous flow theory was emphasized.

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